Physica A 519 (2019) 295–302

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Average hopcount of the shortest path in tree-like components with finite size

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ARTICLE INFO

Article history:
Received 6 June 2017
Received in revised form 14 October 2017
Available online 2 January 2019

MSC:
05C80
05C05

Keywords:
Random graph
Average shortest path
Finite component
Configuration model

ABSTRACT

An exact formula for computing the average hopcount of the shortest path in finite-size tree-like components of undirected unweighted random networks is proposed. In a tree-like component with size s, there exists virtually only one shortest path between two arbitrary nodes. The summation of hopcounts of all shortest paths can be calculated approximately by the summation of the betweenness of all nodes, and the difference between them is only a constant $s(s-1)$. Therefore, the average hopcount can be calculated by further dividing the summation by the number of all shortest paths. In this paper, we first derive the conditional probability $p(k|s)$ of the degree distribution of finite components with size s and the summation of all nodal betweenness respectively. By means of these results, we obtain the exact formula for calculating the average hopcount. Also, We confirm the proposed formula by simulations for networks with Poisson and power law degree distributions respectively.

1. Introduction

Random networks have been used to characterize the statistical properties of many real networking systems such as World Wide Web (WWW) [1], internet and many real-world networks. The research on random networks has been attracting ample attention since the concept of random networks was first proposed by Erdös and Rényi [2]. Over the past decades, some mathematical models such as the configuration model [3,4] have been proposed to be the framework of random graphs. Furthermore, network properties such as the betweenness centrality [5–8], the structurally statistical features of components [9], and the structural robustness [10–13] have been investigated.

One of the important statistical characteristics of random networks is the average shortest path length (ASPL). Specifically, in the undirected unweighted network concerned in this paper, the ASPL is equivalent to the average hopcount of the shortest path between two arbitrary nodes. Usually, a random graph is composed of one giant component and many finite-size components [4,9]. Considering the concept of ASPL is defined for connected networks, previous works put emphasis mainly on the statistical properties of the shortest path in the giant component. Many theoretical works have been proposed to calculate the ASPL given that all nodes are reachable from any selected node. Newman et al. [4] argued that the ASPL of the giant component scales as $\log_\nu N$, where N is the size of the giant component, $\nu = g_0''(1)$ and $g_0(z)$ denotes the generating function of the degree distribution. This argument was soon confirmed for random graphs with various degree

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https://doi.org/10.1016/j.physa.2018.12.034
0378-4371/

distributions under the frameworks of variants of the configuration model [14–19]. By means of the framework of random graphs named the hidden variable model [20], Fronczak et al. [21] also proposed an exact formula for ASPL in the network with any degree distribution, although the good agreement between theoretical predictions and numerical results only exist in dense networks. The theoretical research regarding the shortest path also has many applications such as understanding the functional brain networks [22,23].

On the other hand, previous works usually assume that all vertices are reachable from any given node. However, this will not be true, especially for graphs with no giant component. Even if there exists the giant component, it is not likely that it fills the entire graph. In order to gain thoroughly understanding of the whole topological structure, it appears necessary to investigate the statistical properties (e.g. the ASPL) of finite-size components in the undirected unweighted random graph. Newman investigated the component size distribution for random graphs with given degree distribution [9]. He et al. proposed an exact formula to calculate the nodal betweenness in finite-size components [7]. Besides, the statistical properties (e.g. the ASPL and the component size distribution) of many real networks have also been investigated [24]. However, previous work has paid little attention on the statistical property ASPL in finite-size components.

This paper is organized as follows. In Section 2, we first derive the general formula to calculate the average hopcount in the finite-size component with s nodes, which employ as parameters the degree distribution of the finite-size component and the average nodal betweenness. As the average nodal betweenness has been proposed by the previous work, we turn to deriving the formula of the degree distribution of the finite-size component and expressing it in the form of the degree distribution of the whole random graph. Finally, we apply the general average hopcount formula to the random graphs with Poisson degree distribution and power-law degree distribution. In Section 3, we verify the proposed formula for Poisson and power-law distributions respectively and investigate the asymptotic behavior of the formula with respect to the component size s. In Section 4, we conclude the paper.

2. Average hopcount in finite-size components

2.1. Definitions

The network generating model employed in this paper is the configuration model which is also known as the framework of the random graph with prescribed degree sequence [3]. The method for generating networks by this model is depicted as follows. We first generate a degree sequence following an expected degree distribution $p(k)$. Then we randomly connect two vertices by an undirected edge. By repeating the above connection process until no vertices are left, we finally generate a network with the desired degree distribution. Many properties of networks generated by the configuration model have been derived, such as the average path length of the giant component and the distribution of finite-size components [3,4]. The connectedness of an entire random graph has a critical point, or saying a phase transition, where a giant component forms and contains a rather large portion of nodes in the graph. Other than the giant component, there always exist many fragmented finite-size components of different sizes. Even for the finite components of the same size, the structures of them are manifold. In this paper, we aim to propose a formula computing the expected value of the average shortest path length of all finite-size components of size s where s is the parameter.

Suppose that we pick up a finite component of size of s from a network randomly generated by the configuration model. We intend to compute the average hopcount of the shortest path of this component. Because finite components are almost surely trees [4,25], there exists virtually only one shortest path between a specific pair of nodes and consequently there are just $s(s-1)$ shortest paths in the component. Since the concerned network in this paper is undirected and unweighted, the number of the edges a shortest path traverses equals the hopcount of the shortest path. Therefore, the summation of hopcounts of all shortest paths equals to the summation of the betweenness of all nodes. By dividing this summation by $N(N-1)$, we finally gain the expected value of the hopcount of a shortest path.

2.2. Theoretical deduction: the nodal betweenness based method

Let $p(k)$ be the degree distribution of the entire random network. Then, the mean value of the degree can be expressed as

$$\langle k\rangle = \sum_k kp(k). \tag{1}$$

Randomly choosing an edge and following it to one of the vertex it connects, the number of other edges emerging from that vertex follows the so-called excess degree distribution:

$$q(k) = \frac{(k+1)p(k+1)}{\langle k\rangle}, \tag{2}$$

where $\langle k\rangle = \sum_k kp(k)$ is the average degree, as shown in [4]. Let $g_0(z)$ and $g_1(z)$ be the generating function of $p(k)$ and $q(k)$ respectively

$$g_0(z) = \sum_k p(k)z^k, \tag{3}$$

$$g_1(z) = \sum_k q(k)z^k. \tag{4}$$

According to [4], we obtain the following relations

$$g_1(z) = \frac{g_0'(z)}{g_0'(1)}, \tag{5}$$

$$\langle k \rangle = g_0'(1). \tag{6}$$

In terms of the above introduced generating functions (3) and (4) than in terms of degree distributions $p(k)$ and (2), our derived results in this paper can be expressed in a more elegant way.

Denote by s, H_i, $\bar{H} = E[H]$, $p(k|s)$ and $\langle b(s,k)\rangle$ the size of the finite component concerned, the length (hopcount) of the ith shortest path (there are $s(s-1)$ shortest paths in the component), the expected value of hopcount of all shortest paths, the probability that the degree of a randomly selected node is k in a component of size s, the expected value of the betweenness of a node of degree k in a component of size s. The shortest path of length H_i contributes $L_i - 1$ to the summation of the betweenness of all nodes, while to the summation of the betweenness of all edges it adds H_i. Therefore,

$$\sum_{i=1}^{s(s-1)} (H_i - 1) = 2s \cdot \sum_{k=1}^{s-1} p\,(k|s)\,\langle b(s,k)\rangle$$

$$\Rightarrow \sum_{i=1}^{s(s-1)} L_i - s(s-1) = 2s \cdot \sum_{k=1}^{s-1} p\,(k|s)\,\langle b(s,k)\rangle$$

$$\Rightarrow \left(\bar{H} - 1\right) s(s-1) = 2s \cdot \sum_{k=1}^{s-1} p\,(k|s)\,\langle b(s,k)\rangle$$

$$\Rightarrow \bar{H} = \frac{2}{s-1} \sum_{k=1}^{s-1} p\,(k|s)\,\langle b(s,k)\rangle + 1. \tag{7}$$

Next, we need to derive $\langle b(s,k)\rangle$ and $p\,(k|s)$ respectively. As

$$\langle b(s,k)\rangle = \left\langle \sum_{i\neq j} n_i n_j \right\rangle, \tag{8}$$

has been derived by He et al. [7], the left problem is to derive $p\,(k|s)$.

According to Bayes' theorem, we can rewrite the conditional probability $p\,(k|s)$ as

$$p(k|s) = \frac{p(s|k)p(k)}{p(s)}, \tag{9}$$

$$\langle k \rangle_s = kp(k|s), \tag{10}$$

where $p(k|s)$ denotes the probability of a randomly selected node being degree k in a component of size s, $\langle k\rangle_s$ is the mean degree of the finite-size component, $p(s|k)$ is the probability that a vertex of degree k belongs to a component of size s, and $p(s)$ is the probability that a randomly selected vertex belongs to a component of size s. The exact form of $p(s)$ as a function of the generating function $g_1(z)$ has been derived by Newman [9] and can be expressed as

$$p(s) = \frac{\langle k \rangle}{(s-1)!} \left[\frac{d^{s-2}}{dz^{s-2}} \left[g_1(z)\right]^s \right]_{z=0}. \tag{11}$$

Then we need to derive $p(s|k)$ in (9). Denote by t and ρ_t the number of vertices reachable through a specific edge and the probability distribution of t. Let the generating function for ρ_t be $h_1(z) = \sum_t \rho_t z^t$. As defined in [9], the generating function $h_0(z)$ for $p(s)$ is

$$\begin{aligned} h_0(z) &= \sum_{s=1}^{\infty} p(s) z^s \\ &= \sum_{s=1}^{\infty} \sum_{k=0}^{\infty} p(k)p(s|k)z^s \\ &= \sum_{k=0}^{\infty} p(k) \sum_{s=1}^{\infty} p(s|k)z^s \\ &= z \sum_{k=0}^{\infty} p(k)[h_1(z)]^k. \end{aligned} \tag{12}$$

Observing the last step of the above deduction, it is straightforward to obtain an expression of the generating function $h(z)$ of $p(s|k)$, i.e.,

$$h(z) = \sum_{s=1}^{\infty} p(s|k) z^s = z[h_1(z)]^k. \tag{13}$$

Applying the properties of generating function [4] and the definition (13), we can express $p(s|k)$ as

$$\begin{aligned} p(s|k) &= \frac{1}{s!}\frac{d^s}{dz^s}[h(z)]_{z=0} \\ &= \frac{1}{s!}\frac{d^s}{dz^s}\left[z\,[h_1(z)]^k\right]_{z=0} \\ &= \frac{1}{s!}\frac{d^{s-1}}{dz^{s-1}}\left[[h_1(z)]^k + z\frac{d}{dz}[h_1(z)]^k\right]_{z=0}. \end{aligned} \tag{14}$$

The expression (14) can be further rewritten by means of Cauchy's formula for the nth derivative of a function,

$$\frac{d^n}{dz^n}[f(z)]\bigg|_{z=z_0} = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}dz. \tag{15}$$

Applying (15) to (14), analogous to the deduction in [9], we can further obtain the conditional probability

$$\begin{aligned} p(s|k) &= \frac{1}{s!}\left\{\frac{d^{s-1}}{dz^{s-1}}\left[[h_1(z)]^k\right]_{z=0} + (s-1)\frac{(s-2)!}{2\pi i}\oint \frac{\frac{d}{dz}[h_1(z)]^k}{z^{s-1}}dz\right\} \\ &= \frac{1}{s!}\left\{\frac{d^{s-1}}{dz^{s-1}}\left[[h_1(z)]^k\right]_{z=0} + (s-1)\frac{d^{s-1}}{dz^{s-1}}\left[[h_1(z)]^k\right]_{z=0}\right\}. \end{aligned} \tag{16}$$

Applying the relation between $h_1(z)$ and $g_1(z)$

$$\left[\frac{d^n}{dz^n}[h_1(z)]^m\right]_{z=0} = \frac{m(n-1)!}{(n-m)!}\left[\frac{d^{n-m}}{dz^{n-m}}[g_1(z)]^n\right]_{z=0}, \tag{17}$$

which has been derived in [7], to (16), we gain the following relation

$$\begin{aligned} p(s|k) &= \frac{1}{(s-1)!}\left\{\frac{k(s-2)!}{(s-1-k)!}\left[\frac{d^{s-1-k}}{dz^{s-1-k}}[g_1(z)]^{s-1}\right]_{z=0}\right\} \\ &= \frac{k}{(s-1)(s-1-k)!}\left[\frac{d^{s-1-k}}{dz^{s-1-k}}[g_1(z)^{s-1}]\right]_{z=0}. \end{aligned} \tag{18}$$

By means of (11) and (18), we can derive (9). Eventually, with (7) we can calculate the average hopcount of the shortest paths $\bar{H}$ of a finite component with s nodes.

2.3. Applications in Poisson and power-law degree distributions

Take the Poisson degree distribution network for example, we obtain the following results

$$p(k) = \mathrm{e}^{-c}\frac{c^k}{k!},\; c = \langle k\rangle, \tag{19}$$

$$g_1(z) = \mathrm{e}^{c(z-1)},\; \left[\frac{d^n}{dz^n}[g_1(z)]^m\right]_{z=0} = (mc)^n \mathrm{e}^{-mc}, \tag{20}$$

$$p(s) = \frac{\mathrm{e}^{-cs}(cs)^{s-1}}{s!}, \tag{21}$$

$$p(s|k) = \frac{k\,(c(s-1))^{s-1-k}\,\mathrm{e}^{-c(s-1)}}{(s-1)(s-1-k)!}, \tag{22}$$

$$p(k|s) = \frac{s!(s-1)^{s-2-k}}{(k-1)!s^{s-1}(s-1-k)!}. \tag{23}$$

Taking (23) into (9) and combining with the analytical form of $\langle b(s,k)\rangle$ [7], we could calculate exactly the average hopcount (7).

A great number of studies have revealed that degree distributions of many real networks such as world wide web (WWW) [26] and internet [27] have a power-law tail, taking the form

$$p(k) \propto k^{-\alpha} \tag{24}$$

for some constant exponent α. The exponent α usually lies in the range $2 < \alpha < 3$. Without loss of generality, we take the exponent value 2.5 for α. As in [9], we employ the so-called Yule distribution for the excess degree distribution $q(k)$ which asymptotically takes a power law form with the power exponent $\alpha - 1$ for large k and corresponds to the power law degree distribution $p(k)$ with exponent α,

$$q(k) = C_0 \frac{\Gamma(k + 1/2)}{\Gamma(k + 2)}, k \geq 0. \tag{25}$$

The generating function $g_1(z)$ of $q(k)$ and the derivative of powers of $g_1(z)$ derived by He et al. [7] take the form

$$g_1(z) = \frac{1}{1 + \sqrt{1 - z}}, \tag{26}$$

$$\left[\frac{d^n}{dz^n}[g_1(z)]^m\right]_{z=0} = m2^{-(m+2n)}\frac{(m + 2n - 1)!}{(m + n)!}. \tag{27}$$

where C_0 is a normalized factor needed to be determined. Substituting (27) into (18) yields

$$p(s|k) = k2^{3+2k-3s}\frac{(3s - 2k - 4)!}{(s - k - 1)!(2s - k - 2)!}. \tag{28}$$

It is also convenient to obtain the corresponding degree distribution $p(k)$, its generating function $g_0(z)$, the average degree $\langle k \rangle$, and the normalized factor C_0 in (25) (see Appendix for the deduction),

$$p(k + 1) = \frac{q(k)\langle k \rangle}{k + 1} = \frac{\langle k \rangle\, \Gamma(k + 1/2)}{2\sqrt{\pi}(k + 1)\Gamma(k + 2)}, k \geq 0, \tag{29}$$

$$g_0(z) = \frac{\ln\left(1 + \sqrt{1 - x}\right) - \sqrt{1 - x}}{1 - \ln 2} + 1, \tag{30}$$

$$\langle k \rangle = \frac{1}{2\,(1 - \ln 2)}, \tag{31}$$

$$C_0 = \frac{1}{(2\sqrt{2})}. \tag{32}$$

Substituting the component size distribution derived by Newman [9]

$$p(s) = [1 - \ln 2]^{-1}\, s2^{3-3s}\frac{(3s - 5)!}{(s - 1)!(2s - 2)!}, \tag{33}$$

the degree distribution (29) and (28) into (9) yields $p(k|s)$ for Yule distribution. Analogous to the Poisson case above mentioned, combining with the analytical form of $\langle b(s, k)\rangle$ derived in [7], we could calculate exactly the average hopcount (7).

3. Simulation results

Given a specific degree distribution $P(k)$, we first generate 10^4 networks with each network having 10^4 nodes. Next, we focus on the finite components extracted from the generated networks. We classify finite components according to their sizes, e.g., the number s of nodes in a finite component. For each kind of the finite component, we calculate the average shortest path length $\bar{H}$.

We first verify the proposed formula (7) for Poisson case with the average degree $c = 1.2$. As shown in Fig. 1, the theoretical results numerically calculated by (7) fit the simulations well. Also, the proposed formula fits simulations well for networks with Yule excess degree distribution as illustrated in Fig. 1. The discrepancies between the theoretical curve and the simulations for large s come from the experimental statistical deviation. The probability that a finite-size component does not belong to the giant component decrease exponentially as the size number s increases. Thus, the number of samples of finite-size components obtained is dramatically small, which cause some statistical deviation.

In addition, we investigate the asymptotic behavior of the average hopcount (7) of the shortest path as a function of the size s of the finite component. The log–log plots of $\bar{H}$ versus s are shown in Fig. 2 for the Poisson case and the power-law case. It seems that the average hopcount versus s follows a linear relation in the log–log plot.

4. Conclusion

In this paper, we proposed an exact formula to calculate the average hopcount of the shortest path in finite tree-like components of random networks with arbitrary degree distributions. The average hopcount is expressed in terms of the nodal betweenness and the degree distribution of finite-size components. We made effort to get the exact form of the degree distribution of finite-size components. Incorporating with the expression of the nodal betweenness obtained by previous work, we obtained the exact formula to calculate the average hopcount and used two degree distributions, Poisson

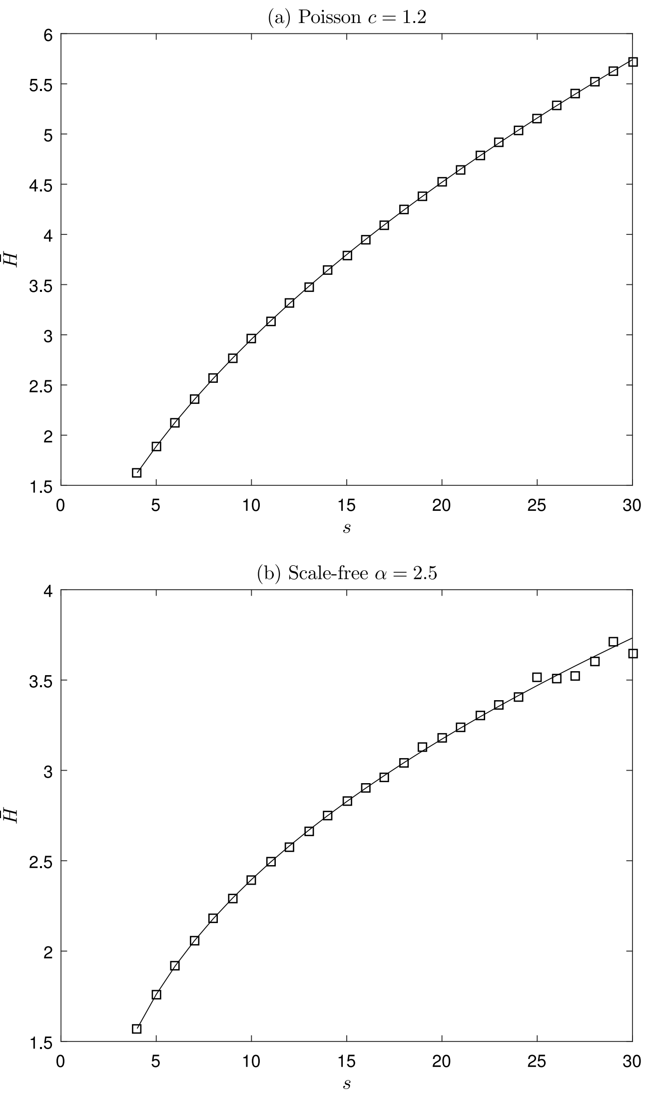

Fig. 1. Comparison between the numerically calculated formula (7) and the simulation results on the average shortest path length versus the component size s for networks with (a) Poisson degree distribution with average degree $c = 1.2$, and (b) Yule excess degree distribution. The solid curve and scattered squares denote theoretical results and simulation results respectively.

and power law, to validate the formula through simulations. Besides, we analyzed the asymptotic behavior of the average hopcount and revealed the linear relation between the average hopcount and the component size in the log–log axis. The results we obtained hold exactly in trees and may benefit those who concern the study regarding the percolation and the topological structure of random networks. The findings of this paper certainly helps to get better understanding of the finite-size components. Based on the findings, one may further investigate the topological robustness of finite-size components with respect to random failures or intentional attacks. Besides, in the study of temporal networks, a small time scale may cause a network with finite-size components. The findings obtained in this paper in finite-size components could helpful to understand the properties of temporal networks.

Acknowledgments

We would like to thank the anonymous reviewers for their comments and suggestions. We also thank Shan He for his valuable suggestions. This work was funded in part by the National Natural Science Foundation of China under Grant No. 61603097, No. 71731004, No. 61672318, No. 61631013, and No. 61702439, in part by the National Key Research and Development Program of China under Grant No. 2016YFB1000102, in part by the Natural Science Foundation of Shanghai under Grant No. 16ZR1446400, in part by the European Seventh Framework Program (FP7) under Grant No. PIRSES-GA-2012-318939, and in part by the Fundamental Research Funds (No. 021014380099) for the Central Universities.

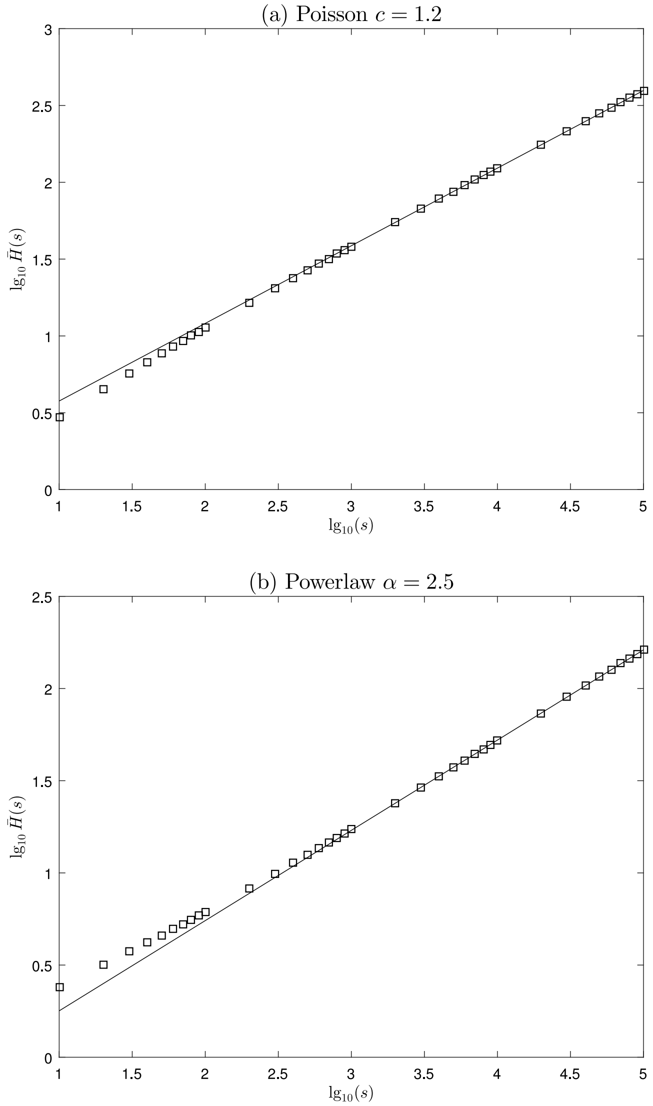

Fig. 2. Log–log plots of $\bar{H}$ as a function of s. The numerically calculated formula (7) versus s are shown for networks with (a) Poisson degree distribution with average degree $c = 1.2$, and (b) Yule excess degree distribution. The solid curve and scattered squares denote curve fitting results and simulation results respectively.

Appendix. Yule distribution

Newman et al. [9] derived the generating function (26) of the excess degree distribution (25) of Yule distribution. For the sake of simplicity, we denote by $C_1 = g_0'(1) = \langle k \rangle$ the average degree (6). Using (5), we get

$$\begin{aligned} g_0(z) &= C_1 \int g_1(z) dz \\ &= C_1 \int \frac{1 - \sqrt{1-z}}{1 - \left(\sqrt{1-x}\right)^2} dz \\ &= C_1 \int \frac{1 - \sqrt{1-z}}{z} dz \\ &= C_1 \left(\int \frac{1}{z} dz - \int \frac{\sqrt{1-z}}{z} dz \right). \end{aligned} \quad \text{(A.1)}$$

Letting $t = \sqrt{1-z}$, we can rewrite (A.1)

$$\begin{aligned} g_0(z) &= C_1\left(\ln|z| + \int \frac{2t^2}{1-t^2}dt\right) + C_2 \\ &= C_1\left(\ln|z| + 2\left(\int \frac{1}{1-t^2}dt - \int 1dt\right)\right) + C_2 \\ &= C_1\left(\ln|z| + \ln\left|\frac{1+\sqrt{1-z}}{1-\sqrt{1-z}}\right| - 2\sqrt{1-z}\right) + C_2 \\ &= C_1\left(\ln\left(1+\sqrt{1-z}\right)^2 - 2\sqrt{1-z}\right) + C_2. \end{aligned} \tag{A.2}$$

For the probability function $p(k)$ is assumed to be normalized, we obtain $g_0(1) = 1$. Besides, for $k \geq 1$, it is convenient to note that $g_0(0) = 0$. Using $g_0(1) = 1$, $g_0(0) = 0$ and (A.2) yields $C_1 = 1/(2 - 2\ln 2)$ and $C_2 = 1$. So, we get the average degree (31).

Next, we aim to derive the exact form of the normalized factor C_0 of (25). Using (2) for $k = 0$, it is convenient to get

$$p(1) = \langle k\rangle\, q(0) = \frac{C_0\sqrt{\pi}}{(2-2\ln 2)}. \tag{A.3}$$

Using the property of generating function and (5), we get

$$p(1) = \left[\frac{d}{dz}g_0(z)\right]_{z=0} = g_1(0)g_0'(1) = \frac{1}{4(1-\ln 2)}. \tag{A.4}$$

Combining (A.3) and (A.4) yields the normalized factor (32).

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